ROTATING INCOMPRESSIBLE PERFECT FLUID SOURCE OF THE NUT METRIC

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A global solution of the Einstein equations is given, consisting of a perfect fluid interior and a vacuum exterior. The rigidly rotating and incompressible perfect fluid is matched along the hypersurface of vanishing pressure with the stationary part of the Taub-NUT metric. The fluid core generates a negative-mass NUT space-time. The matching procedure leaves one parameter in the global solution.

1 Introduction

Finding examples of space-times consisting of a rotating core and vacuum exterior is one of the many difficult issues in general relativity. A recent discussion on the matching of an axisymmetric rotating interior fluid with ambient vacuum by Mars and Senovilla ¹ reached the conclusion that the conditions of continuity of the first and second fundamental forms represents an overdetermined system.

In this paper we construct a space-time containing a rotating perfect fluid surrounded by vacuum. We join a rigidly rotating incompressible perfect fluid solution ^{2,3,4} with the Taub-NUT metric ^{5,6}, motivated by the isomorphic Killing algebras and by the peculiarities in the causal behavior of both space-times. What we get, however, is not what one would ideally expect since the fluid core does not rotate in the usual sense because it is locally rotationally symmetric (LRS).

The perfect fluid space-time has been described in detail elsewhere ⁴. We cut this space-time along the hypersurface of zero pressure, which we make the common boundary with the vacuum region. We then apply the Darmois-Israel matching procedure ^{8,9}. The continuity of both the induced metric and extrinsic curvature lead to severe restrictions on the parameters of the two solutions: all of them but one are frozen at some specific values. This is still a fortunate result since there are more equations to fulfill than parameters.

Our signature convention is (+---) and we choose Einstein's gravitational constant $8\pi G = 1$.

2 Joining the two metrics

The interior solution was found by Ferwagner² and later rediscovered by Marklund³. In a recent work⁴ it has been cast in a simple form with its properties further analyzed. This perfect fluid metric is:

$$ds^{2} = \sin^{4} \chi (dt + 2R\cos\theta d\varphi)^{2} - 2R\sin^{2} \chi d\chi (dt + 2R\cos\theta d\varphi)$$
$$- R^{2}\sin^{2} \chi (d\theta^{2} + \sin^{2}\theta d\varphi^{2}), \qquad (1)$$

where R is the only parameter of the solution. The density is constant, $\mu = 6/R^2$, and the fluid is rigidly rotating with the four-velocity

$$u = \sin^{-2} \chi \frac{\partial}{\partial t} \ . \tag{2}$$

The pressure depends on the radial variable χ as

$$p = \frac{4}{R^2} \sin^{-2} \chi - \frac{6}{R^2} \,. \tag{3}$$

The condition of vanishing pressure selects a hypersurface $\chi = \chi_1$ defined by

$$\sin \chi_1 = \sqrt{\frac{2}{3}}.\tag{4}$$

At $\chi < \chi_1$ the pressure is positive, diverging in the origin, where the metric is also singular. We cut the space-time along the embedding hypersurface $\chi = \chi_1$ keeping the part where $\chi < \chi_1$.

The Taub-NUT metric has become renowned for being 'a counterexample to almost anything' 7 . The extension of the originally homogeneous Taub metric 5 is done by introducing new coordinates, obtaining a space-time in which homogeneous and stationary parts are glued along Cauchy horizons. In these coordinates the Taub-NUT metric is

$$ds^{2} = \frac{r^{2} - 2mr - l^{2}}{r^{2} + l^{2}} (d\tau + 2l\cos\theta d\varphi)^{2} - 2dr(d\tau + 2l\cos\theta d\varphi)$$
$$- (r^{2} + l^{2}) (d\theta^{2} + \sin^{2}\theta d\varphi^{2})$$
(5)

with the parameters m and l. The domain lying between the roots $r_{\pm} = m \pm \sqrt{m^2 + l^2}$ of the numerator of the $g_{\tau\tau}$ metric coefficient describes a homogeneous cosmology. Outside the roots the metric (5) is stationary.

The striking similarity in the appearence of the two metrics (1) and (5) allows us to immediately list the conditions for continuity of the induced metrics

at the junction hypersurface $\chi = \chi_1$ and $r = r_1$. These conditions are:

$$r_1^2 + l^2 = R^2 \sin^2 \chi_1 \tag{6}$$

$$l = \alpha R \tag{7}$$

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 (7)

$$\alpha^2 \frac{r_1^2 - 2mr_1 - l^2}{r_1^2 + l^2} = \sin^4 \chi_1.$$
 (8)

At the junction the angles θ and φ of the two metrics are identified and the relation $\tau = \alpha t$ is assumed. Thus altogether there are four parameters in the global space-time: α, m, l and R. From (4) and (6) the coordinate r_1 at the junction is expressed in terms of the other parameters:

$$r_1 = R\sqrt{\frac{2}{3} - \alpha^2},\tag{9}$$

while (8) can be regarded as the defining equation for the parameter m in terms of the rest of the parameters.

Let us denote the space-time coordinates by $x^a = (t, \rho, \theta, \varphi)$ where ρ denotes the radial coordinate, χ or r. The coordinates $\xi^{\alpha} = (t, \theta, \varphi)$ can be chosen as coordinates in both embeddings of the junction hypersurface. A holonomic basis is thus given by $e^a_{(\alpha)} = \partial x^a/\partial \xi^\alpha = \delta^a_\alpha$. The extrinsic curvature is

$$K_{\alpha\beta} = e^a_{(\alpha)} e^b_{(\beta)} \nabla_a n_b = -\Gamma^1_{\alpha\beta} (-g^{11})^{-1/2}.$$
 (10)

The last equality applies because the normal to the junction hypersurface is $n_a = (g^{11})^{-1/2} \delta_a^1$ for both metrics, g^{11} being $-1/R^2$ for the fluid metric and $-\left(r^2 - 2mr - l^2\right)/\left(r^2 + l^2\right)$ for the Taub-NUT metric.

The condition to have no jump in the extrinsic curvature at the junction, thus to have no distributional matter shell separating the two domains, after some algebra and employing the relations (7), (8) and (9), translates to:

$$\alpha = \frac{2}{3}, \qquad l = \frac{2}{3}R, \qquad m = -\frac{2\sqrt{2}}{3}R.$$
 (11)

Thus all but one parameters are fixed, We end up with a Universe which has a single parameter R. The mass parameter turns out to be negative.

The Cauchy horizons $r_{\pm} = 2R \left(-\sqrt{2} + \sqrt{3}\right)/3 < r_1 = R\sqrt{2}/3$ of the Taub-NUT solution lie on the fluid side of the junction hypersurface, thus they are not present in the global space-time. Hence we have a fluid interior joined with a stationary region of the Taub-NUT space-time.

The causal behaviour of the global solution we have obtained is of interest. In both domains, at the junction hypersurface, the vector $e^a_{(\varphi)}$ becomes null at $\tan \theta_{\pm} = \pm 2\sqrt{2/3}$ and timelike at $\theta \in [0, \theta_{+}) \cup (\theta_{-}, \pi]$. This implies the existence of closed timelike curves, a familiar feature of the Taub-NUT spacetime.

The above discussion was generalized recently ¹⁰ for a class of LRS perfect fluid space-times, all of them being sources for the NUT metric.

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